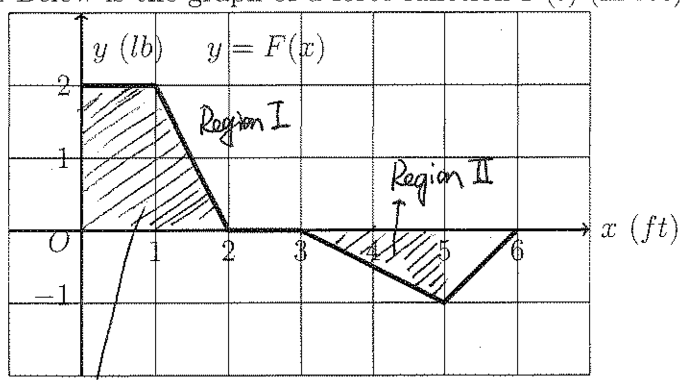
Solutions: (For Hints and Explanation, please refer to the previous Hint-Document).

Practice Midterm1 (Take-home quiz 6, due on Tue, Oct 11)

Q1 Below is the graph of a force function $F(t)$ (in lbs).



(a) How much work is done by the force in moving an object from $x = 0$ to $x = 3$?

Area of Region I: $W = \frac{1}{2} \times 2 \times (1+2) = \boxed{3 \text{ ft-lb}}$. Remark: the work done from $x=2$ to $x=3$ is ZERO.

(b) How much work is done by the force in moving an object from $x = 0$ to $x = 5$?

Area of Region II: $\frac{1}{2} \times 1 \times 2 = 1$

W (from 0 to 5) = Area of Region I − Area of Region II $= 3 - 1 = \boxed{2 \text{ ft-lb}}$

Q2 Find the derivatives of the following functions

(a) $f(x) = [\tan^{-1} x]^{\ln(2x)}$, find $f'(x)$

$y = [\tan^{-1}x]^{\ln 2x}$

$\ln y = \ln(\tan^{-1}x^{\ln 2x})$

$\ln y = \ln(2x) \cdot \ln(\tan^{-1}x)$

Take derivative w.r.t. x.

$\frac{1}{y} \cdot y' = \frac{1}{x} \cdot \ln(\tan^{-1}x) + \ln(2x) \cdot \frac{1}{\tan^{-1}x} \cdot \frac{1}{1+x^2}$ product rule + chain rule.

$$\boxed{y' = [\tan^{-1}x]^{\ln(2x)} \cdot \left[\frac{\ln(\tan^{-1}x)}{x} + \frac{\ln(2x)}{\tan^{-1}x \cdot (1+x^2)}\right]}$$

(b) $f(x) = \pi^{\sin^{-1}(\sqrt{x})}$, find $f'(x)$

$f'(x) = \ln\pi \cdot \pi^{\sin^{-1}(\sqrt{x})} \cdot \frac{d}{dx}\sin^{-1}(\sqrt{x})$, $(a^x)' = \ln a \cdot a^x$

$= \ln\pi \cdot \pi^{\sin^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}\sqrt{x}$, $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

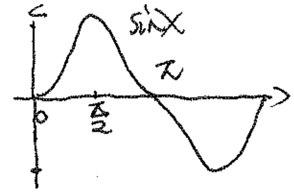
$= \ln\pi \cdot \pi^{\sin^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$.

(c) $f(x) = x^2 + 3\sin x + 1$, find $(f^{-1})'(1)$

$f(0) = 0 + 0 + 1 \Rightarrow f^{-1}(1) = 0$.

$f'(x) = 2x + 3\cdot\cos x$.

$$(f^{-1})'(1) = \frac{1}{f'[f^{-1}(1)]} = \frac{1}{f'(0)} = \frac{1}{2\cdot 0 + 3\cos 0} = \boxed{\frac{1}{3}}$$



Q3 Determine whether the following limits exist or not. Find the limit if it exists.

(a)

$$\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos(2\theta)}$$

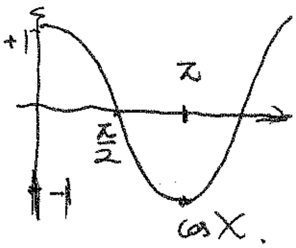
$\overset{l'H}{=} \lim_{\theta\to\frac{\pi}{2}} \frac{(1-\sin\theta)'}{(1+\cos 2\theta)'}$

$= \lim_{\theta\to\frac{\pi}{2}} \frac{-\cos\theta}{-\sin(2\theta)\cdot 2}$ (still $\frac{0}{0}$ type)

$\overset{\text{2nd } l'H}{=} \lim_{\theta\to\frac{\pi}{2}} \frac{(\cos\theta)'}{(2\sin 2\theta)'}$

$= \lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta}{4\cos 2\theta}$

$= \frac{-\sin\frac{\pi}{2}}{4\cos 2\cdot\frac{\pi}{2}} = \frac{-1}{4\cdot(-1)} = \boxed{\frac{1}{4}}$



$\sin\frac{\pi}{2}=1$, $\cos\pi=-1$.

(b)

$$\lim_{x\to+\infty}\sqrt{x}\cdot e^{-x/2}$$

$= \lim_{x\to+\infty} \frac{\sqrt{x}}{e^{\frac{x}{2}}}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{l'H}{=} \lim_{x\to+\infty} \frac{\frac{1}{2\sqrt{x}}}{e^{\frac{x}{2}}\cdot\frac{1}{2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\cdot e^{\frac{x}{2}}} = \frac{1}{\infty\cdot\infty} = \boxed{0}$

(c)

$$\lim_{t\to 0^+} t\cdot\tan^{-1}(1/t)$$

$\lim_{t\to 0^+} \tan^{-1}\left(\frac{1}{t}\right) = \tan^{-1}\left(\frac{1}{0^+}\right) = \tan^{-1}(+\infty) = +\frac{\pi}{2}$

$= 0\cdot\frac{\pi}{2} = \boxed{0}$

Q4 Evaluate the following integrals

(a)

$$\int\sin(3t)\cdot(2t+1)\mathrm{d}t$$

IBP: $u=2t+1$, $dv=\sin(3t)\cdot dt$
$du=2$, $v=-\frac{1}{3}\cos(3t)$

$= u\cdot v - \int v\cdot du$

$= (2t+1)\left[-\frac{1}{3}\cos(3t)\right] - \int -\frac{1}{3}\cos 3t\cdot dt = -\frac{2t+1}{3}\cdot\cos(3t) + \frac{1}{3}\cdot\int\cos(3t)\cdot dt$

$= \boxed{-\frac{2t+1}{3}\cdot\cos(3t) + \frac{1}{3}\cdot\frac{1}{3}\cdot\sin(3t) + C}$

(b)

$$\int_0^\infty 2^{-x}\mathrm{d}x$$

$= \lim_{t\to\infty}\int_0^t (2^{-1})^x dx = \lim_{t\to\infty} \left.\frac{(2^{-1})^x}{\ln 2^{-1}}\right|_0^t = \lim_{t\to\infty} \frac{2^{-t}}{\ln 2^{-1}} - \frac{1}{\ln 2^{-1}}$

Hint: $\lim_{t\to\infty} 2^{-t} = \lim_{t\to\infty}\frac{1}{2^t} = 0$

$\boxed{= -\frac{1}{\ln 2^{-1}} = \frac{1}{\ln 2}.}$

Hint: $\ln 2^{-1} = (-1)\cdot\ln 2$.

(c)

$$\int\frac{5}{\sqrt{9-25x^2}}\mathrm{d}x$$

u-sub: $9-25x^2 = 3^2-(5x)^2$

$= \int\frac{5}{\sqrt{9-(3u)^2}}\cdot\frac{3}{5}du$ $\boxed{5x=3u}$, $dx=\frac{3}{5}du$

$= \int\frac{3\,du}{\sqrt{9(1-u^2)}} = \int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u)+C = \boxed{\sin^{-1}\left(\frac{5x}{3}\right)+C}$

Q4 Evaluate the following integrals.

(a)

$$\int \frac{x^2}{(x^2+1)^{5/2}}\,\mathrm{d}x$$

$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$.

$\boxed{x^2+1 = \tan^2\theta + 1 = \sec^2\theta.}$

$$= \int \frac{\tan^2\theta}{[\sec^2\theta]^{\frac{5}{2}}} \cdot \sec^2\theta \cdot d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^5\theta} \cdot \sec^2\theta \cdot d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

$$= \int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^3\theta}}\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta \cdot d\theta$$

$$= \int \sin^2\theta \cdot \underline{\cos\theta \cdot d\theta}$$

$u = \sin\theta$
$du = \cos\theta \cdot d\theta$

$$= \int u^2 \cdot du$$

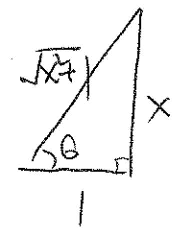
$$= \frac{1}{3} \cdot u^3 + C$$

$$= \frac{1}{3} \cdot \sin^3\theta + C.$$

$$= \boxed{\frac{1}{3} \cdot \left[\frac{x}{\sqrt{x^2+1}}\right]^3 + C}$$

$\tan\theta = x = \frac{x}{1}$

$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+1}}$



(b)

$$\int_8^\infty \frac{10}{x^2-4x-21}\,\mathrm{d}x$$

$x^2 - 4x - 21 = (x-7)\cdot(x+3)$.

P.F.D. $\frac{10}{(x-7)(x+3)} = \frac{A}{x-7} + \frac{B}{x+3} = \frac{1}{x-7} - \frac{1}{x+3}$.

$\Rightarrow 10 = A(x+3) + B(x-7)$

$x=7 \Rightarrow 10 = A\cdot 10 \Rightarrow A = 1$

$x=-3 \Rightarrow 10 = B\cdot(-10) \Rightarrow B = -1$.

$$= \lim_{t\to\infty} \int_8^t \frac{1}{x-7} - \frac{1}{x+3}\, dx$$

$$= \lim_{t\to\infty} \left[\ln|x-7| - \ln|x+3|\right]\Big|_8^t$$

$$= \lim_{t\to\infty} \left[\ln|t-7| - \ln|t+3|\right] - \left[\ln 1 - \ln 11\right]$$

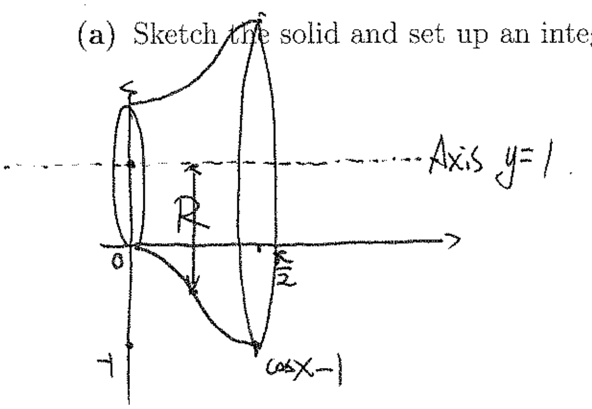
$$= \lim_{t\to\infty} \ln\left|\frac{t-7}{t+3}\right| + \ln 11$$

$$= \ln \lim_{t\to\infty} \frac{|t-7|}{|t+3|} + \ln 11$$

$$= \ln 1 + \ln 11 = \boxed{\ln 11}$$

Q5 The solid is generated by revolving the curve $y = \cos x - 1$ for $0 \le x \le \pi/2$ about the axis $y = 1$.

(a) Sketch the solid and set up an integral for the volume of it.



$R = 1 - [\cos x - 1] = 2 - \cos x.$

$A(x) = \pi \cdot R^2 = \pi \cdot [1 - (\cos x - 1)]^2 = \pi \cdot [2 - \cos x]^2$

$V = \int_0^{\frac{\pi}{2}} A(x)\,dx = \pi \cdot \int_0^{\frac{\pi}{2}} (2 - \cos x)^2\,dx.$

(b) Find the volume of the rotating solid.

$V = \pi \cdot \int_0^{\frac{\pi}{2}} (2 - \cos x)^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} 4 - 4\cos x + \cos^2 x\,dx.$

$= \pi \int_0^{\frac{\pi}{2}} 4 - 4\cos x + \frac{1 + \cos 2x}{2}\,dx.$

$= \pi \int_0^{\frac{\pi}{2}} \frac{9}{2} - 4\cos x + \frac{1}{2}\cos 2x\,dx$

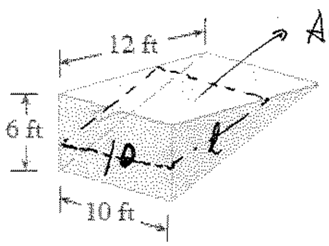
$= \pi \cdot \left[\frac{9}{2}x - 4\sin x + \frac{1}{2} \cdot \frac{1}{2}\sin 2x\right]\Big|_0^{\frac{\pi}{2}}$

$= \pi \left[\frac{9}{2} \cdot \frac{\pi}{2} - 4\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi\right] - \pi[0 - 0 + 0]$

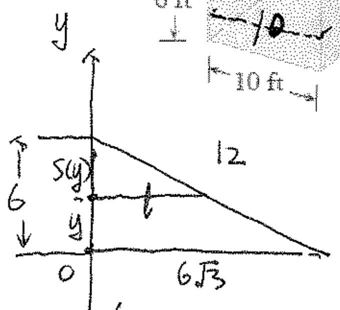
$= \pi \cdot \left[\frac{9}{4}\pi - 4 + 0\right]$, $\sin\frac{\pi}{2} = 1$, $\sin\pi = 0$

$= \boxed{\pi \cdot \left(\frac{9}{4}\pi - 4\right)}$ (or $\frac{9}{4}\pi^2 - 4\pi$).

Q6 A tank (shown below) is full of oil weighing 10 lb/ft^3. Find the work required to pump the oil out of the spout. The base is a $10 \times 6\sqrt{3}$ rectangle. The back end is a 6×10 rectangle, the two sides are right triangle with height 6, base $6\sqrt{3}$ and hypotenuse 12 (all in ft). The spout is 6 ft from the base.



$A(y) = 10 \cdot l.$



$\frac{l}{6\sqrt{3}} = \frac{6 - y}{6} \Rightarrow l = \sqrt{3}(6 - y)$

$W = \int_0^6 \delta \cdot s(y) \cdot A(y)\,dy$

$\delta = 10$

$s(y) = 6 - y$

$A(y) = 10 \cdot l. = 10 \cdot \sqrt{3}(6 - y)$

$W = \int_0^6 10 \cdot (6 - y) \cdot 10\sqrt{3} \cdot (6 - y)\,dy$

$= 100\sqrt{3} \int_0^6 (6 - y)^2\,dy$

$= 100\sqrt{3} \cdot \int_0^6 36 - 12y + y^2\,dy$

$= 100\sqrt{3} \cdot \left[36y - 6y^2 + \frac{1}{3}y^3\right]\Big|_0^6$

$= 100\sqrt{3} \cdot \left[36 \cdot 6 - 6 \cdot 6^2 + \frac{1}{3} \cdot 6^3\right]$

$= 100\sqrt{3} \cdot \cancel{\frac{1}{3}} \cdot \cancel{6 \cdot 6} \cdot 6$ 72

$= \boxed{7200\sqrt{3} \text{ ft-lb}}$


6-y
L
y
6√3


6-y
L
L/(6-y) = 6√3/6
y
or L/(6√3) = (6-y)/6
6√3


6-y
L
y
L/(6-y) = 6√3/6
or L/(6√3) = (6-y)/6
6√3

Build your OWN!

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

Rotating case:

$A(x)=\pi\cdot[f(x)-\text{AXIS}]^2$

$$V=\int_a^b A(x)\,dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

Water-Pumping:

$W=\int_a^b \sigma\cdot s(y)\cdot A(y)\,dy$

$$W=\int_a^b f(x)\,dx$$

$(\sqrt{x})'=\frac{1}{2\sqrt{x}}$

$(\frac{1}{x})'=-\frac{1}{x^2}$

- $\int \frac{1}{x}\,dx=\ln|x|+C$, $\int x^n dx=\frac{1}{n+1}x^{n+1}+C$ $n\neq -1$
- $\int \tan x\,dx=\ln|\sec x|+C$, $(\tan x)'=\sec^2 x$, $(\cot x)'=-\csc^2 x$

$(\sec x)'=\tan x\cdot\sec x$, $(\csc x)'=-\cot x\cdot\csc x$.

- $\int \sec x\,dx=\ln|\sec x+\tan x|+C$

$(a^x)'=\ln a\cdot a^x$

- $\int a^x\,dx=\frac{a^x}{\ln a}+C$ for $a\neq 1$

$a^x=e^{\ln a\cdot x}$

$a^{-x}=\frac{1}{a^x}$, $a^x=\frac{1}{a^{-x}}$.

- **Integration by Parts**:

$$\int u\,dv=uv-\int v\,du$$

$\int \underbrace{\text{Poly}}_{u}\times\underbrace{\exp/\sin x/\cos x\,dx}_{dv}$

$dv=e^{ax}dx \Rightarrow v=\int e^{ax}dx=\frac{1}{a}e^{ax}$

$dv=\sin(ax)dx \Rightarrow v=\int \sin(ax)dx=-\frac{1}{a}\cos(ax)$

$dv=\cos(ax)dx \Rightarrow v=\int \cos(ax)dx=\frac{1}{a}\sin(ax)$

- limits: $e^{+\infty}=+\infty$, $e^{-\infty}=0$, $\ln\infty=\infty$, $\ln 0^+=-\infty$

$\tan^{-1}(\pm\infty)=\pm\frac{\pi}{2}$. $e^0=1$, $\ln 1=0$

- L'Hopital Rule: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

$\frac{\infty}{\infty}$, $\frac{0}{0}$, $0\cdot\infty=\frac{0}{\frac{1}{\infty}}=\frac{\infty}{\frac{1}{0}}$

- P.F.D. $\frac{*}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$, $\frac{*}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$

$\int\frac{1}{x-a}dx=\ln|x-a|$, $\int\frac{1}{(x-a)^2}dx=-\frac{1}{x-a}$.

Derivatives

- $\frac{d}{dx}(\sinh x)=\cosh x$ $\quad$ $\frac{d}{dx}(\cosh x)=\sinh x$
- Inverse Trigonometric Functions:

$\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x$, $\int\frac{dx}{\sqrt{a^2-b^2x^2}}$, u-Sub: $bx=au$

$$\boxed{\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}} \qquad \frac{d}{dx}(\csc^{-1}x)=\frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}} \qquad \frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$$

$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}(\frac{x}{a})$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a))\neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

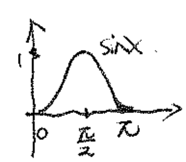
- Hyperbolic Functions

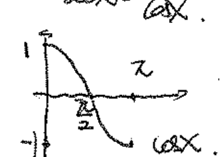
$$\sinh(x)=\frac{e^x-e^{-x}}{2} \qquad \operatorname{csch}(x)=\frac{1}{\sinh x}$$

$$\cosh(x)=\frac{e^x+e^{-x}}{2} \qquad \operatorname{sech}(x)=\frac{1}{\cosh x}$$

$$\tanh(x)=\frac{\sinh x}{\cosh x} \qquad \coth(x)=\frac{\cosh x}{\sinh x}$$

- $\cosh^2 x-\sinh^2 x=1$
- $\boxed{\cos^2 x+\sin^2 x=1}$ $\quad\tan^2 x+1=\sec^2 x$, $\tan x=\frac{\sin x}{\cos x}$, $\sec x=\frac{1}{\cos x}$.
- $\sin^2 x=\frac{1}{2}(1-\cos 2x)$
- $\cos^2 x=\frac{1}{2}(1+\cos 2x)$
- $\sin(2x)=2\sin x\cos x$





- $\sin A\cos B=\frac{1}{2}[\sin(A-B)+\sin(A+B)]$
- $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$
- $\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$
- Trig-Sub: $\sqrt{a^2-b^2x^2}$ $\quad bx=a\sin\theta$

$\sqrt{a^2+b^2x^2}$ $\quad bx=a\tan\theta$

$\sqrt{b^2x^2-a^2}$ $\quad bx=a\sec\theta$.